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LETTER TO THE EDITOR

Driving force on an Abrikosov vortex

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Abstract
The standard result for the force per unit length on a vortex is derived for an arbitrary configuration of vortices and transport currents in the London limit. The sign reversal between this force, and the Lorentz force on a current in a magnetic field, is shown to be because of the fact that the vortex and the currents driving it are embedded in a single condensate.

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The standard driving force on a vortex in a superconductor due to other supercurrents has the form

$$\mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) \times \mathbf{\Phi}_0 \tag{1}$$

where $\mathbf{F}(\mathbf{r})$ is the force per unit length on a vortex at $\mathbf{r}$, $\mathbf{J}(\mathbf{r})$ is the current density at $\mathbf{r}$ due to all sources *other than the vortex itself*, and the vector $\mathbf{\Phi}_0$ is equal in magnitude to the flux quantum $\phi_0 = hc/(2e)$ and points in the direction of the magnetic flux in the vortex. $\mathbf{F}$ in equation (1) is often stated to be the Lorentz force on the vortex due to the interaction between the current $\mathbf{J}$ and the vortex magnetic field, but this is actually not the case, see equation (8), most importantly because the sign is wrong. In this letter we discuss the physical origin of equation (1) in detail.

Equation (1) is usually derived [1, 2] by computing the force between two long parallel vortices or a vortex and a boundary, and generalizing the result. Although the generalization is a very natural one, since a vortex cannot 'know' the origin of the supercurrents in its vicinity, this is based on the assumption that the force only depends on local quantities. This is not so obvious: as we shall see in the second half of this letter, the fact that both the vortices are embedded in the same condensate is important in understanding why equation (1) differs from the Lorentz force of classical electromagnetism, equation (8). But the existence of an extended condensate opens the possibility of a *non-local* form for the vortex force. In fact, the local form in equation (1), in which the force at $\mathbf{r}$ only depends on the current density at $\mathbf{r}$, is correct, but it is worth investigating the reason for this. Furthermore, if the supercurrent $\mathbf{J}$ were a transport current, driven by external sources, one would have to include the work done by the external source when the vortex moves in order to obtain the force from energy conservation. This is not an issue for the intervortex interaction, and so one might be concerned that the expression for the force could depend on whether the current is due to external sources or other vortices, which is actually not the case. It is thus useful to obtain equation (1) under general circumstances and to clarify its physical origin.

Using the London free energy [1, 2][1],

$$\mathcal{F} = \frac{1}{2}\int \mathrm{d}\tau_I\, \lambda^2 J^2 + \frac{1}{2}\int \mathrm{d}\tau\, B^2 \tag{2}$$

where the first integral runs over the superconductor and the second over all space, the change in free energy when the vortex moves a small distance is

$$\delta\mathcal{F} = \int \mathrm{d}\tau_I\, \lambda^2 \mathbf{J}\cdot\delta\mathbf{J} + \int \mathrm{d}\tau\, \mathbf{B}\cdot\delta\mathbf{B}. \tag{3}$$

In calculating the force on the vortex from energy balance, we also have to consider the work done by external sources. If the current I is injected into the superconductor at $\mathbf{r}_1$ and extracted at $\mathbf{r}_2$ (it is easy to generalize this to multiple sources and sinks), the work done by the external source driving the current when the vortex moves is

$$\delta W = \int \mathrm{d}\tau_E\, \mathbf{J}\cdot\delta\mathbf{A} - I(\hbar/2e)[\delta\chi_1 - \delta\chi_2]. \tag{4}$$

The first term is from the Faraday emf, and is an integral over the region outside the superconductor (including the external circuit path from $\mathbf{r}_2$ to $\mathbf{r}_1$). The second term comes from the potential difference between the two points due to the vortex motion, $(\hbar/2e)[\partial_t\chi_1 - \partial_t\chi_2]$, where χ is the phase of the superconducting order parameter [3]. Therefore

$$\delta\mathcal{F} - \delta W = \int \mathrm{d}\tau_I\, \lambda^2\mathbf{J}\cdot\delta\mathbf{J} + \int \mathrm{d}\tau\, \mathbf{B}\cdot\delta\mathbf{B} - \int \mathrm{d}\tau_E\, \mathbf{J}\cdot\delta\mathbf{A} + I(\hbar/2e)(\delta\chi_1 - \delta\chi_2). \tag{5}$$

Integrating by parts, $\int \mathrm{d}\tau\, \mathbf{B}\cdot\delta\mathbf{B} = \int \mathrm{d}\tau\, \mathbf{J}\cdot\delta\mathbf{A}$. This cancels $-\int_E \mathbf{J}\cdot\delta\mathbf{A}$, leaving $\int \mathrm{d}\tau_I\, \mathbf{J}\cdot\delta\mathbf{A}$. Also, $I(\delta\chi_1 - \delta\chi_2) = \int \mathrm{d}\tau_E\, \mathbf{J}\cdot\nabla(\delta\chi)$ is equal to $-\int \mathrm{d}\tau_I\, \mathbf{J}\cdot\nabla(\delta\chi)$. Expressing $\lambda^2\mathbf{J}$ as $(\hbar/2e)\nabla\chi - \mathbf{A}$ and simplifying equation (5) yields

$$\delta\mathcal{F} - \delta W = (\hbar/2e)\int \mathrm{d}\tau'\, \mathbf{J}\cdot[\delta(\nabla\chi) - \nabla(\delta\chi)]. \tag{6}$$

This is not zero because of the singularity in $\nabla\times\nabla\chi$. For a displacement $\delta\mathbf{r}$ of the vortex,

$$\delta(\nabla\chi) - \nabla(\delta\chi) = 2\pi(\hat{n}\times\delta\mathbf{r})\delta^2(\mathbf{r}) \tag{7}$$

where $\hat{n}$ is the (local) direction of the vortex and $\delta^2(\mathbf{r})$ is non-zero at the vortex core. Thus $\delta\mathcal{F}_{\mathrm{int}} - \delta W = -\Phi_0\int[\mathbf{J}\times \mathrm{d}l]\cdot(\delta\mathbf{r})$, with a line integral along the vortex core. This yields equation (1) for the force per unit length on the vortex.

There are a few special cases worth noting here. Firstly, when the currents flow entirely in the superconductor, either due to other vortices or as closed current loops. There is no external work done in this case, and the driving force on the vortex comes from the change in $\mathcal{F}_{\mathrm{int}}$. Secondly, when the driving currents are entirely outside the superconductor, from equation (6) there is no force on the vortex[2]. Thus the same current flowing on or just off the surface of a superconductor gives rise to different forces on the vortex, unlike the Lorentz force in classical electromagnetism. Finally, for a stream of vortices flowing through a superconducting sample in steady state, $\mathcal{F}_{\mathrm{int}}$ and $\mathbf{A}$ are constant, and the force on the vortices comes solely from

[1] SI units are used throughout this paper, but with $\mu_0 = 1$.

[2] Except for indirect forces, if the external currents induce a screening current on the superconductor.

the external work done by the source of transport current against the electrostatic potential difference $(\hbar/2e)(\partial_t\chi_1 - \partial_t\chi_2)$.

We next consider *why* the force on a vortex differs from the Lorentz force in classical electromagnetism. Since the Lorentz force on a current due to a magnetic field is $\int \mathrm{d}\tau\, \mathbf{J}\times\mathbf{B}$, the force on the vortex due to the supercurrent $\mathbf{J}$ should have been its opposite:

$$\mathbf{F}_L = -\int \mathrm{d}\tau(\mathbf{J}\times\mathbf{B}_\mathbf{v}). \tag{8}$$

This differs from equation (1) by a sign, and by not being concentrated at the vortex core. Since equation (8) is valid for all magnetic materials (and for electrical circuits), this leads to the question: why does a superconductor behave differently from all other magnetic materials?

In order to understand the difference, we look for the closest analogue with magnets for a system of vortices. Accordingly, we consider the case of two vortices with no external currents in a superconductor, compared to two weakly coupled magnets whose magnetism is entirely orbital in origin. In the first case, the interaction part of the free energy is

$$\mathcal{F}_{\mathrm{int}} = \int \mathrm{d}\tau[\lambda^2\mathbf{J}_1\cdot\mathbf{J}_2 + \mathbf{B}_1\cdot\mathbf{B}_2] \tag{9}$$

where $\mathbf{J}_{1,2}$ are the currents due to the two vortices, and $\mathbf{B}_{1,2}$ are the corresponding magnetic fields. In the second case, the free energy consists of the material free energy,

$$\mathcal{F}_{\mathrm{mat}} = \sum_{1,2}\frac{1}{2m}\int \mathrm{d}\tau|(-\mathrm{i}\hbar\nabla - e\mathbf{A})\psi|^2 + \text{potential energy} \tag{10}$$

and the magnetic field energy. For weak coupling, the vector potential $\mathbf{A}_2$ due to the second magnet is a small perturbation on the first (and vice versa), and the interaction part of the free energy is

$$\mathcal{F}_{\mathrm{int}} = \int \mathrm{d}\tau[-\mathbf{J}_1\cdot\mathbf{A}_2 - \mathbf{J}_2\cdot\mathbf{A}_1 + \mathbf{B}_1\cdot\mathbf{B}_2]. \tag{11}$$

There is also a change in $\psi_{1,2}$ because of $\mathbf{A}_{2,1}$ but the effect on $\mathcal{F}$ from this is second order.

Equation (11) differs from equation (9) by (i) the existence of two 'non-magnetic' terms instead of one, (ii) $\mathbf{J}_{1,2}$ being coupled to $\mathbf{A}_{2,1}$ instead of $\mathbf{J}_{2,1}$. Both of these differences come from the existence of two separate wavefunctions for the two magnets, whereas the two vortices are embedded in a single condensate. Thus the difference between the two equations is because the vortices are not really separate superposable objects, a somewhat surprising conclusion since the linearity of the London equations apparently justifies treating vortices as separate and superposable. Standard manipulations on equation (11) show that the effect of the third term is reversed by the first two, yielding equation (8); on the other hand, in equation (9) the first term concentrates the intervortex interaction at the core without reversing its sign, as in equation (1)[3].

Although we have made the comparison using orbital magnetism, because the resemblance between equations (9) and (11) brings out their essential difference, equation (8) is of course general for magnets. On thermodynamic grounds, one can define the magnetization density $\mathbf{M}$ of a magnet by the condition $\delta\mathcal{F}_{\mathrm{mat}} = -\int \mathrm{d}\tau\mathbf{M}\cdot\delta\mathbf{B}$ for an arbitrary change $\delta\mathbf{B}$ in the magnetic field[4]. If one now considers two magnets and moves the second one infinitesimally

[3] For a superconductor that is not strongly type-II, the Ginzburg Landau free energy can yield more complicated intervortex interactions [4].

[4] That this definition is correct can be seen by varying the total free energy—including the magnetic term—with respect to $\mathbf{B}$: stationarity with respect to any divergence free $\delta\mathbf{B}$ requires that $\mathbf{B}-\mathbf{M}$ should be curl free, which is equivalent to the standard definition of $\mathbf{M}$ in the absence of external currents, $\nabla\times\mathbf{B} = \nabla\times\mathbf{M}$. This defines $\mathbf{M}$ up to the gradient of a scalar potential. For a superconductor, this is all one can achieve; the local magnetization has no physical significance beyond this (see [5], section 53). This is similar to the situation with polarization in conductors.

towards the first, simple integration by parts yields $\delta\mathcal{F} = \delta\mathcal{F}_B + \delta\mathcal{F}^1_{\mathrm{mat}} + \delta\mathcal{F}^2_{\mathrm{mat}} = -\delta\mathcal{F}_B$, since the second and third terms in the intermediate expression are equal and opposite to the first. Implicit in this derivation is the assumption that the magnets can be treated as separate objects, whose magnetization can be varied independently.

We note that when one is not considering the interaction of two distinct magnetized objects, equation (8) breaks down even without superconductivity. For example, consider a material which has a magnetic phase with magnetization density $\mathbf{M}$ and a non-magnetic phase. Let the free energy densities in the first and the second phases in the absence of an applied magnetic field be $f_M(0)$ and $f_N(0)$ respectively. For simplicity, consider a long vertical sample in a vertical applied magnetic field. If the applied magnetic field (not the actual magnetic field in the sample) is increased from $\mathbf{B}_0$ to $\mathbf{B}_0 + \delta\mathbf{B}_0$, the change in $f_N - f_M$ is [5] equal to $\mathbf{M}\cdot\delta\mathbf{B}_0$. Phase coexistence occurs at the value of B_0 for which $f_N(B_0) - f_M(B_0) = 0$, i.e. the value of B_0 for which

$$f_N(0) - f_M(0) = -\int \mathrm{d}\mathbf{B}_0 \cdot \mathbf{M}(\mathbf{B}_0). \tag{12}$$

On the other hand, with a vertical boundary between the two phases, the Lorentz force per unit area that would have been exerted on the boundary can be obtained from the surface current M and the magnetic fields $\mathbf{B}_0$ and $\mathbf{B}_0 + \mathbf{M}$ on the two sides of the boundary. The Lorentz force per unit area would have been $-\mathbf{M}\cdot\left(\mathbf{B}_0 + \frac{1}{2}\mathbf{M}\right)$. From equation (12), this is equal to the 'condensate' pressure $f_N(0) - f_M(0)$ only if $\mathbf{M}\cdot\left(\mathbf{B}_0 + \frac{1}{2}\mathbf{M}\right) = \int \mathrm{d}\mathbf{B}_0\cdot\mathbf{M}(\mathbf{B}_0)$. Integrating the right-hand side by parts, this is equivalent to the condition

$$\int \mathrm{d}\mathbf{M}\cdot(\mathbf{B}_0 + \mathbf{M}) = 0. \tag{13}$$

While this is true when the magnetic phase is perfectly diamagnetic, it is not true in general. This is because equation (8) was derived for (rigid) translations of magnets, and not for a change in shape of a single magnetized region. The unusual feature of the intervortex interaction is that it involves translations of apparently independent objects and still violates equation (8).

It has been argued [6] that the difference between equations (1) and (8) is because the force on a vortex is primarily not a magnetic force at all, but rather a 'dynamic quantum force'. This is based on the observations that the energy of a single vortex is dominated by its kinetic energy (the first term in equation (2)), and that equation (1) can be expressed as a current–current interaction. Although these observations are correct, the conclusion is erroneous. The Lorentz force in magnets is *also* predominantly 'dynamical quantum' in origin. The field energy by itself would cause two parallel vortices to repel—since the field energy is quadratic in the magnetic field—like equation (1), but without concentrating the force at the vortex core:

$$\mathbf{F}_f = \int \mathrm{d}\tau(\mathbf{J}\times\mathbf{B}). \tag{14}$$

As we have seen, in ordinary magnetic materials $\mathcal{F}_{\mathrm{mat}}$ gives rise to an additional force twice as large and opposite to equation (14), yielding equation (8). These extra terms are quantum in origin for spin magnetism, and dynamic and quantum for orbital magnetism: the first part of $\mathcal{F}_{\mathrm{mat}}$ in equation (10) is the direct analogue of the first part of $\mathcal{F}$ in equation (2). In fact quantum effects are *less* important in superconductors, since equation (1) only concentrates the force of equation (14) at the vortex core, while equation (8) reverses it[5].

[5] For superfluids, or—as [6] remarks—Josephson junction networks, the force on a vortex is entirely from the kinetic energy of the superfluid, but this is not true for bulk superconductors.

To conclude, we have obtained the standard expression for the force on an Abrikosov vortex in a superconductor, regardless of the origin of the supercurrents in its vicinity, and explained its form intuitively. In particular, we have shown that the force on a vortex from another vortex is not the same as for ordinary magnetic materials, because a *single* condensate runs through the entire system.

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